CS6999

Probabilistic Methods in Integer Programming

Randomized Rounding

Andrew D. Smith

April 2003

Background

- â Integer Programming
- â The Probabilistic Method
- â Randomization and Approximation
- â The Covering Radius of a Code

Randomized Rounding

- â The Algorithm
- â Probabilistic Analysis

Handling Feasibility

- â Covering and Packing Problems
- â Scaling Variables

Derandomization

- â Conditional Probabilities
- â Pessimistic Estimators

Advanced Techniques

- â Positive Correlation
- â Limited Dependence

maximizing or minimizing a linear function over a polyhedron

given matrix $A \in \mathbb{R}^{m \times n}$
vector $b \in \mathbb{R}^m$
vector $c \in \mathbb{R}^n$

find a vector x optimizing

$$\max\{cx | Ax \leq b\}$$

$$\min\{cx | x \geq 0; Ax \geq b\}$$

Alexander Schrijver (1986)
"Theory of linear and integer programming"

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x is the solution vector

$\max cx$ or $\min cx$ are objective functions

$Ax \leq b$, $Ax \geq b$, $x \geq 0$ are constraints on the solution

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linear programming	$x \in \mathbb{R}^n$	polynomial time
integer programming	$x \in \mathbb{Z}^n$	NP-complete (decision version)

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solving optimization problems over discrete structures

famous integer programs

- Facility Location
- Knapsack Problem
- Job Shop Scheduling
- Travelling Salesperson

all NP-hard!

some exact solution techniques

- Branch and Bound
- Dynamic Programming
- Cutting Planes
- Ellispoid/Separation

exact algorithms fail eventually!

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exact algorithms fail eventually!

what happens when no exact algorithm is fast enough

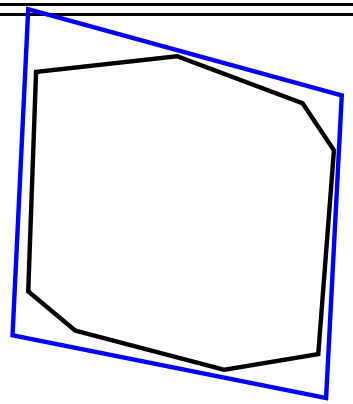
but some solution must be found?

Find an approximately optimal solution

more on this shortly . . .

relaxation: remove constraints

geometrically:	expand (and simplify) the polyhedron
translation:	allow more vectors to be solutions



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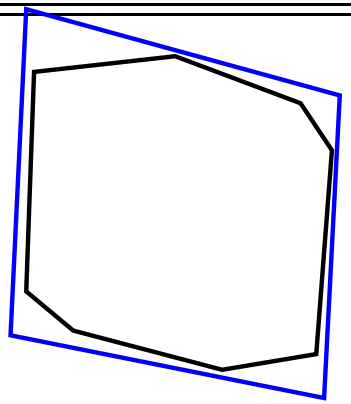
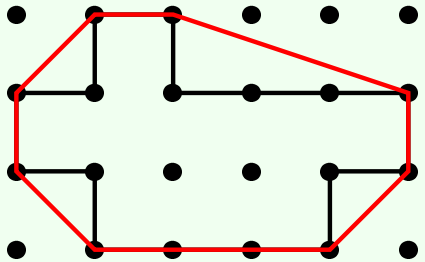


fractional relaxation: remove integrality constraints

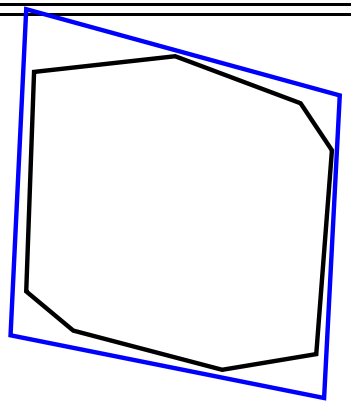
very important technique in solving integer programs

$$x \in \mathbb{Z}^n \longmapsto \bar{x} \in \mathbb{R}^n$$



- recall that linear programs can be solved in polynomial time (e.g. primal dual interior point)
- fractional relaxations for NP-hard problems can be very easy (e.g. fractional knapsack reduces to sorting)
- provide intuition about (geometric?) structure of problem
- (important concept) integrality gap $= c\bar{x}/cx$

relaxation: remove constraints



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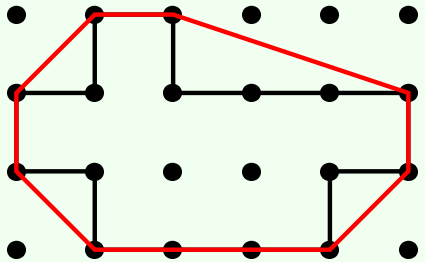

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true understanding of an NP-complete compinatorial optimization problem requires understand its fractional relaxation

Randomized Algorithms

any algorithm with behavior depending on random bits (we assume a source of random bits)

quality measures
- time complexity
- number of random bits required
- probability of success

Motwani & Raghavan (1995) “Randomized algorithms”

D. Hochbaum (1995) “Approximation algorithms for NP-hard problems”

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Approximation Algorithms

polynomial time algorithm for an NP-hard optimization problem

performance ratio $\frac{\text{optimal}}{\text{approximate}}$ (maximization) $\frac{\text{approximate}}{\text{optimal}}$ (minimization)

performance ratios are better when closer to 1 and cannot be < 1

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combine the two Randomized Approximation Algorithms

Motwani & Raghavan (1995) “Randomized algorithms”
D. Hochbaum (1995) “Approximation algorithms for NP-hard problems”

> Trying to prove that a structure with certain desired properties exists,
> one defines an appropriate probability space of structures
> and then shows that the desired properties hold in this space
> with positive probability. — Alon & Spencer

Alon & Spencer (2000) “The probabilistic method” 2nd edition

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importance for algorithms

- can be used to show a solution exists
- and to show that a solution exists with particular qualities
- and to show that a solution exists where an algorithm will find it
- essential tool in design and analysis of randomized algorithms

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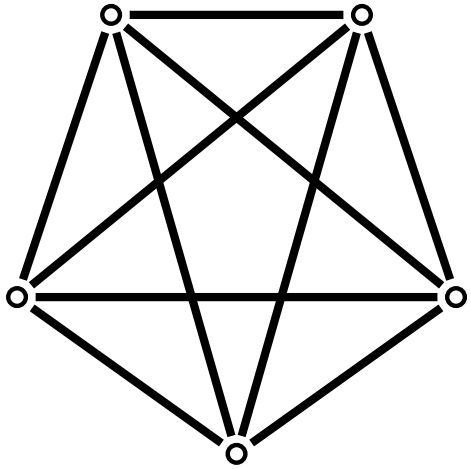
caveat non-constructive by definition

Alon & Spencer (2000) “The probabilistic method” 2nd edition

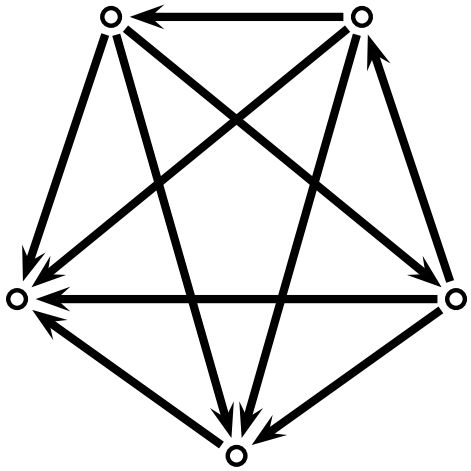
K_5 is the complete graph on 5 vertices



A Quick Illustration

K_5 is the complete graph on 5 vertices

A 5 player tournament is
an orientation of K_5

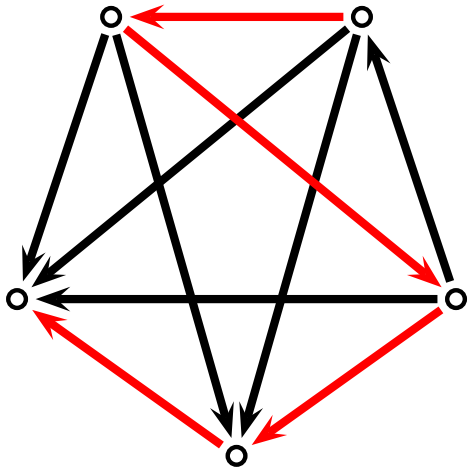


A Quick Illustration

K_5 is the complete graph on 5 vertices

A 5 player tournament is
an orientation of K_5

A Hamiltonian path touches
each vertex exactly once

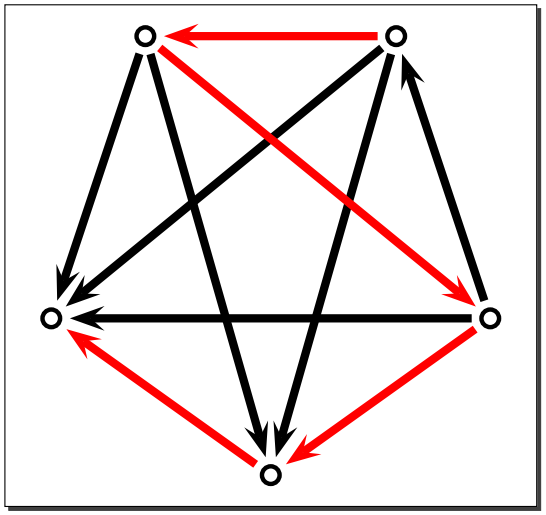


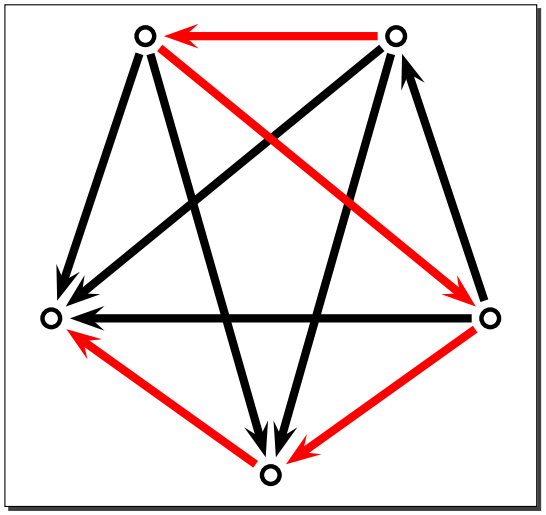
Theorem

There is a tournament with n players and at least

$$n!2^{(1-n)}$$

Hamiltonian paths.





Theorem

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Proof

Let T be a random tournament on n players and let X be the number of Hamiltonian paths in T.

For each permutation σ let X_σ be the indicator random variable for σ giving a Hamiltonian path - i.e., satisfying $(\sigma(i), \sigma(i+1)) \in T$ for $1 \leq i \leq n$.

Then $X = \sum X_\sigma$ and

$$\mathrm{E}[X] = \sum \mathrm{E}[X_\sigma] = n!2^{(1-n)}.$$

Thus some tournament has at least $\mathrm{E}[X]$ Hamiltonian paths.

formal problem statement

instance: a code $\mathcal{F} \subseteq \{0,1\}^n$, $\mathcal{F} = \{s_1, \ldots, s_m\}$.

objective: find a vector z minimizing $\max_i d_H(s_i, z)$.

- d_H is the Hamming distance (number of mismatching positions)
- the vector z is called the center of $\mathcal{F}$
- special case of the Closest String problem (highly approximable)
- important problem in clustering and classification

$\mathcal{F} = \{s_1, s_2, s_3, s_4\}$

z = 0110111100

s_1 = 0111111110

s_2 = 0000110110

s_3 = 1110111100

s_4 = 1111001111

$d_H(s_4, z) = 6$

Frances & Litman (1997) “On covering problems of codes”

identify vector s_i in $\mathcal{F}$ with row a_i of A

for each position j, row a_i has two positions

$$(s_j = 0) \Leftrightarrow (x_{j_1} = 0) \wedge (x_{j_2} = 1)$$
$$(s_j = 1) \Leftrightarrow (x_{j_1} = 1) \wedge (x_{j_2} = 0)$$

example transformation

$s_1 =$	1	0	1	1	0	1	0	0	1	1
$a_1 =$	10	01	10	10	01	10	01	01	10	10

fractional relaxation

$$\bar{x}_{j_1}, \bar{x}_{j_2} \in [0, 1]$$

minimize d

subject to $\sum_{j=1}^{n} (a_{ij_1} x_{j_1} + a_{ij_2} x_{j_2}) \leq d,$

$$\forall 1 \leq j \leq n, x_{j_1} + x_{j_2} = 1, x_{j_1}, x_{j_2} \in \{0, 1\}$$

$\mathcal{F} = \{s_1, s_2, s_3, s_4\}$ $\quad A = [a_1, a_2, a_3, a_4]^T$

s_1 = 0111111110 $\quad a_1$ = 01101010101010101001

s_2 = 0000110110 $\quad a_2$ = 01010101101001101001

s_3 = 1110111100 $\quad a_3$ = 10101001101010100101

s_4 = 1111001111 $\quad a_4$ = 10101010010110101010

$$a_4 = 10101010010110101010$$
$$x = 10010110010101011010$$
$$\Rightarrow a_4 x = 6$$

input: a fractional vector $\bar{x} \in \mathbb{R}^n$ (a solution to the linear relaxation)
output: a random vector $x \in \mathbb{Z}^n$ distributed according to $\bar{x}$

start with the solution to the linear programming relaxation

RANDOMIZED ROUNDING$(\bar{x})$ {
  for each position j of x
    with probability $\bar{x}_{j_1}$ set $x_{j_1} \leftarrow 1$ and $x_{j_2} \leftarrow 0$
    otherwise set $x_{j_1} \leftarrow 0$ and $x_{j_2} \leftarrow 1$
  return x
}

all solution vectors produced by the algorithm are valid solutions some are just better than others

example suppose $\bar{x}_{1_1} = 0.68$ and $\bar{x}_{1_2} = 0.32$,
then $\Pr\left[x_{1_1} = 1 \wedge x_{1_2} = 0\right] = 0.68$ and $\Pr\left[x_{1_1} = 0 \wedge x_{1_2} = 1\right] = 0.32$

key idea: treat the solution to the linear programming relaxation as a probability distribution

Raghavan & Thompson (1987)
"Randomized rounding: a technique for provably good algorithms and algorithmic proofs"

three simple ideas used in probabilistic proofs

Linearity of Expectation

For any random variables X, Y and any $a, b \in \mathbb{R}$,

$$\mathrm{E}[aX + bY] = a\mathrm{E}[X] + b\mathrm{E}[Y].$$

Boole's inequality

Let $\mathcal{E}$ be *any* set of events, then

$$\Pr[\vee_{E \in \mathcal{E}} E] \leq \sum_{E \in \mathcal{E}} \Pr[E],$$

regardless of any dependencies between the events.

Chernoff's Bound

Let $X = \sum_{j=1}^{n} X_j$, $X_j \in \{0, 1\}$, and $\mu = \mathrm{E}[X]$, then

$$\Pr[X > (1+\epsilon)\mu] < \left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}} \right)^{\mu} < \exp(-\epsilon^2 \mu / 3)$$

What is the expected quality of the solution?

consider a vector s_i from $\mathcal{F}$
what is the expected distance between s_i and z?

consider a row a_i from A
what is the expected value of $a_i x$?

by our transformation from Minimum Covering Radius to the integer program, these questions are equivalent

we can answer these questions because we know the probabilities

since $\bar{x}$ was used as the probability when rounding

$$\begin{aligned}
\mathrm{E}[d_H(s_i, z)] &= \mathrm{E}[a_i x] \\
&= a_i \mathrm{E}[x] \\
&= a_i \langle \mathrm{E}[x_{1_1}], \mathrm{E}[x_{1_2}], \ldots, \mathrm{E}[x_{n_1}], \mathrm{E}[x_{n_2}] \rangle \\
&= a_i \langle \bar{x}_{1_1}, \bar{x}_{1_2}, \ldots, \bar{x}_{n_1}, \bar{x}_{n_2} \rangle \\
&= a_i \bar{x}
\end{aligned}$$

so we expect the randomly rounded solution vector to be of the same quality as the fractional solution, but it could be better or worse

What is the probability that the quality differs significantly from expectation?

the value of $a_i x$ is a sum of $0-1$ random variables Bernoulli random variables

so we can apply Chernoff's Bound

$$\Pr\left[X \geq (1+\epsilon)\mu\right] \leq \exp(-\epsilon^2 \mu/3)$$

$$\Pr\left[a_i x \geq (1+\epsilon) a_i \bar{x}\right] \leq \exp(-\epsilon^2 a_i \bar{x}/3)$$

so now we have a bound on the probability

that the product of x with any individual constraint

exceeds $1+\epsilon$ times expectation

Hagerup & Rub (1989) "A guided tour of chernoff bounds"

How do we consider all rows simultaneously?

define the bad events

$E_i \equiv$ "$a_i x > (1+\epsilon) a_i \bar{x}$"

solution must avoid all E_i

probability of violating at least one constraint

so we need to bound

$\Pr\left(\bigvee_{i=1}^{m} E_i\right)$

Boole's inequality says

$\Pr\left(\bigvee_{i=1}^{m} E_i\right) \leq \sum_{i=1}^{m} \Pr(E_i)$

combining with Chernoff's bound we require

$$\left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)^{a_i \bar{x}} < \frac{1}{2m}$$

for success with probablity $1/2$

$\Pr(E_i) \leq \frac{1}{2m}$

Selecting Parameters

when we cannot bound the objective function from below

$$\epsilon \geq \frac{\ln(m)}{d}$$

$$\Pr(E_i) < \frac{1}{2m}$$

the probability of failure on a given attempt is $< \frac{1}{2}$

conclusion randomized rounding is a randomized $O(\log m)$-approximation algorithm for Minimum Covering Radius.

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conclusion randomized rounding is a randomized $O(\log m)$-approximation algorithm for Minimum Covering Radius.

with a large bound on the objective function

$$d \geq \frac{3\ln(2m)}{\epsilon^2}$$

$$\Pr(E_i) < \frac{1}{2m}$$

the probability of failure on a given attempt is $< \frac{1}{2}$

conclusion for any $\epsilon > 0$ we can derive the number of repetitions required to guarantee a performance ratio of $1 + \epsilon$.

for non-binary solution vectors, we can often use the fractional part as the probability:

Rounding up $\Pr(x_i = \lceil \bar{x}_i \rceil) = \bar{x}_i - \lfloor \bar{x}_i \rfloor$

Rounding down $\Pr(x_i = \lfloor \bar{x}_i \rfloor) = \lceil \bar{x}_i \rceil - \bar{x}_i$

- many NP-hard problems have fractional relaxations that are much easier than general linear programs

Fractional Knapsack reduces to sorting items in non-decreasing order of value/weight

- the performance ratio is based on using the fractional relaxation as a lower bound exact solution is not required

a maximum matching algorithm can be used to obtain a fractional vertex cover that provides a lower bound on the optimal integral solution

The method can be (and has been) applied to a very wide range of problems and it has been shown to work well in practice.

à Akamai uses a variant of randomized rounding for performance based load balancing

for Minimum Covering Radius any solution produced by randomized rounding was valid

most integer programs are more difficult

a packing problem is the integer program $\max\{cx|Ax \leq b\}$
with the matrix and vectors $A, b, c, x \geq 0$

Note unlike the minimum covering radius, packing problems
have constraints that are not trivially satisfied
because the right hand side b is fixed

examples

Travelling Salesman	pack paths subject to degree 2 and cost constraints
Job Shop Scheduling	pack jobs subject to machine load constraints

the point

it does not matter how good the solution is with respect to the objective function
if a constraint $a_i x \leq b_i$ is violated

randomly rounding can lead to violation of a constraint

the solution is to scale the variables used as probabilites, prior to rounding, so that the rounding process is likely to result in a feasible solution.

define the scaled vector $\bar{x}' = \bar{x}/(1+\epsilon)$ and use $\bar{x}'$ instead of $\bar{x}$ as the probabilities when rounding

analysis

$$\Pr(a_i x > (1+\epsilon) a_i \bar{x}') = \Pr(a_i x > a_i \bar{x}) < \exp\left(\frac{\epsilon^2 a_i \bar{x}}{3(1+\epsilon)}\right)$$

as before, the analysis proceeds using linearity of expectation, Chernoff's bound and Boole's inequality

to calculate the performance ratio we minimize ϵ subject to

$$\frac{\epsilon^2}{(1-\epsilon)} > \frac{2\ln(m/4)}{a_i \bar{x}}$$

key idea: multiply the probabilites by some fraction less than 1

Raghavan & Thompson (1987)
"Randoimized rounding: a technique for provably good algorithms and algorithmic proofs"

what is derandomization?

2 kinds of (algorithmic) derandomization

partial derandomization	reducing the number of coin flips (random bits) required often the goal is to produce better pseudorandom number generators

full derandomization	eliminate the use of random bits (no coin flips) sufficient to reduce random bits to $O(\log n)$ for problem of size n the number of sequences of $O(\log n)$ random bits is bounded by a polynomial

remark

complexity theoretic derandomization	proving that certain randomized complexity classes are equivalent to deterministic classes they are known to contain

similar to proving P=NP, but much more plausible, especially given PRIMES $\in$ P

if a particular coordinate is rounded to a specified value,
is the probability of failure still < 1?

with an oracle for the above question, we would not need random choices

define the event
"fail" $\equiv \vee_{i=1}^{m} E_i$

prior to the first random
choice $\Pr(\text{fail}) < 1$

the first decision
requires selecting from
2 possible choices

$$\Pr(\text{fail}) = \frac{\Pr(\text{fail}|x_{1_1} = 1) + \Pr(\text{fail}|x_{1_1} = 0)}{2}$$

$$\Rightarrow \min\{\Pr(\text{fail}|x_{1_1} = 1), \Pr(\text{fail}|x_{1_1} = 0)\} \leq \Pr(\text{fail})$$

if we knew the conditional probability
we could make each choice
so that $\Pr(\text{fail})$ does not increase

$$\Pr(\text{fail}|\text{all choices made}) \in \{0, 1\}$$

since the probability of failure
did not increase, all E_i have been avoided

key idea: use a probability function
to guide a greedy algorithm

Erdos & Selfridge (1973) "On a combinatorial game"

the difficulty with conditional probabilities is computing them efficiently

the idea of a pessimistic estimator was introduced to characterize
functions that can be used in the place of the true conditional probability

3 defining properties

- ä the estimator always bounds the probability of failure
- ä the estimator initially indicates that probability of failure is not 1
- ä the value of teh estimator never indreases as coputation proceeds

known pessimistic estimators are often discovered by carefull examination of the
probabilistic proofs for the randomized algorithm
pessimistic estimators hide in the proofs of bounds like Chernoff's

surprising observation

the deterministic algorithm has performance that is better than
the expected performance of the randomized algorithm

Prabhakar Raghavan (1988)
"Probabilistic construction of deterministic algorithms: approximating packing integer programs"

very active area of research

two probabilistic ideas

- à correlated events (e.g. planarity and hamiltonicity of graphs)
- à events with limited dependence (e.g. coloring sparse graphs)

randomized algorithm — high probability
derandomized algorithm — positive probability

can we show positive probability when high probability does not hold?

let N be a family of subsets of $\{1, \ldots, n\}$

N is monotone decreasing if $(X \in N) \wedge (X' \subseteq X) \Rightarrow (X' \in N)$

N is monotone increasing if $(X \in N) \wedge (X \subseteq X') \Rightarrow (X' \in N)$

Given a set $N = \{1, \ldots, n\}$ and some $p \in [0,1]^n$ suppose we select a random $Y \subseteq N$ by placing each $i \in N$ in Y independently with probability p_i. For any $F \subseteq 2^N$, let $\Pr(F) \equiv \Pr(Y \in F)$. Let $F_1, F_2 \ldots, F_s \subseteq 2^N$ be any sequence of monotone decreasing families.

$$\Pr\left(\bigwedge_{i=1}^{s} F_i\right) \geq \prod_{i=1}^{s} \Pr(F_i)$$

If we assume a symmetric probability space with $\Pr = p$, then $\Pr\left(\bigwedge_{i=1}^{s} F_i\right) \geq p^s$.

Fortuin, Ginibre & Kasteleyn (1971)
"Correlational inequalities for partially ordered sets"

- let N be the set of positions in the solution vector x
- let the families $F_i = \{S \subseteq N : a_i x \leq b\}$
- each family F_i corresponds to the set of solutions avoiding the bad event E_i
- each F_i fits the definition of a monotone decreasing family

for covering and packing the events of constraint satisfaction are positively correlated
we may treat them as though they are at least as probably as independent events

$$\begin{aligned} \Pr(\bigvee_{i=1}^{m} E_i) &\leq 1 - \Pr(\bigwedge_{i=1}^{m} \bar{E}_i) \\ &\leq 1 - \Pr(\bigwedge_{i=1}^{m} F_i) \\ &\leq 1 - p^m \\ &< 1 \end{aligned}$$

$$\Pr\left(\bigvee_{i=1}^{m} X_i > (1+\epsilon)\bar{X}\right) < mp$$

becomes

$$\Pr\left(\bigvee_{i=1}^{m} X_i > (1+\epsilon)\bar{X}\right) < p^m$$

positive probability is too small for a randomized algorithm
conditional probabilities can be used because a pessimistic estimator is known

Srinivasan (1995)
"Improved approximations of packing and covering problems"

two forms of the Lovasz Local Lemma: general and symmetric
the general form is more powerful but difficult to use in algorithms

Lovasz Local Lemma (Symmetric Form)

Let $Y = \{y_1, \ldots, y_m\}$ be events in an arbitrary probability space. Suppose each event y_i is mutually independent of all other events y_j exception of at most d, and $\Pr(y_i) \leq p$ for all $1 \leq i \leq m$. If $ep(d+1) \leq 1$ then

$$\Pr\left(\bigwedge_{i=1}^{m} \bar{y}_i\right) > 0.$$

"e" is the base of the natural logarithm

the symmetric form is easy to apply in existence proofs

bonus most objects shown to exist by the symmetric Lovasz Local Lemma can be found with a general algorithm known as

Ł Beck's hypergraph coloring algorithm

J. Beck (1991) "An algorithmic approach to the lovasz local lemma I"

Erdos & Lovasz (1975)

"Problems and results on 3-chromatic hypergraphs and some related questions"

the Lovasz Local Lemma applied to integer programming

the events of interest are the E_i previously defined

E_i and E_j are dependent $\Leftrightarrow$ a_i and a_j share a variable

example
a_1 and a_3 are dependent

a_1 0000111110000000000000000
a_2 0000000111111000000000000
a_3 0000000000000100111100000

Very high level overview of Randomized Rounding with Beck's Algorithm

1) do basic randomized rounding
2) remove all variables that do not appear in a violated constraint
3) remove all constraints for which all variables have been removed
4) form a graph on the set of remaining constraints with vertices corresponding to constraints and edges between constraints sharing a variable
5) key idea with high probability the connected components in the graph are small and can therefore be treated independently and efficiently

Chi-Jen Lu (1999)
"A deterministic approximation algorithm for a minimax integer programming problem"

Thank Prof. Horton and Prof. Evans for helping me with this course

Lots more, ask about applying these methods to your problems (especially routing problems)